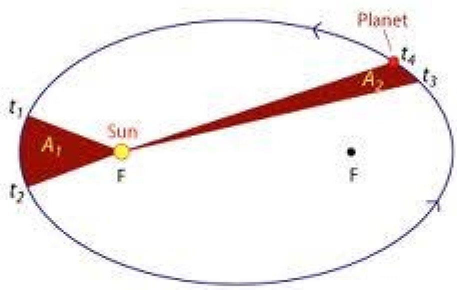
Phys 410
Fall 2015
Lecture #4 Summary
10 September, 2015

As an example of momentum conservation of a many-particle system with non-conservative forces between the particles, we considered a rocket in free space, subject to zero net external force. It can begin to move by ejecting mass at a speed v_{ex} relative to the rocket. By conservation of momentum, the rocket gains an equal and opposite moment to that given to the ejected fuel. While describing the momentum of the rocket + exhaust from an inertial reference frame we found that $m\dot{v} = -v_{ex}\dot{m}$, where v is the speed of the rocket, m is its mass, and $\dot{m}$ is the rate at which it is ejecting mass. The thrust force on the rocket is $-v_{ex}\dot{m}$. We also found an expression for the net change in velocity of the rocket as $v - v_0 = v_{ex} \ln\frac{m_0}{m}$, where m_0 is the initial mass and m is the final mass. In order to maximize the rocket velocity one should maximize the exhaust speed v_{ex} and the ratio $\frac{m_0}{m}$. The exhaust speed typically depends on the violent exothermic chemical reaction that takes place in the rocket motor, making space flight fairly dangerous.

We considered the equations of motion for a system of particles, like a baseball, as opposed to a point particle. We first defined the center of mass of a multi-particle system as $\vec{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha$, the weighted sum of the particle positions, where the total mass of the particles is $M = \sum_{\alpha=1}^{N} m_\alpha$. We can relate the total momentum of the system to the center of mass coordinate as $\vec{P} = M\dot{\vec{R}}$. This shows that we can regard the total momentum of the system of particles as if it were a single particle of mass M moving at the velocity of the center of mass. Further, after taking a time derivative we find that $\dot{\vec{P}} = M\ddot{\vec{R}}$ (which assumes that $\dot{M} = 0$ and that all internal forces obey Newton's third law), which is Newton's second law for the system of particles in terms of the center of mass momentum derivative and acceleration. This equation justifies our frequent treatment of extended objects (like a baseball, satellite, etc.) as point particles that move on a simple trajectory described by Newton's second law of motion. This treatment only considers the translational aspect of the motion of a system of particles. Such systems can also show rotation while they are translating, and this aspect of the motion must also be understood.

To begin to understand rotational motion of a system of particles, consider first the rotation of a single particle about an arbitrarily chosen origin. Angular momentum is a measure of the difficulty of bringing a rotating object to rest. One can define the angular momentum of a single particle, relative to an arbitrarily chosen origin as $\vec{\ell} = \vec{r} \times \vec{p}$, where $\vec{r}$ is the coordinate of the particle and $\vec{p}$ is its linear momentum. We showed that the time-derivative of the angular momentum is $\dot{\vec{\ell}} = \vec{r} \times \vec{F} = \vec{\Gamma}$, where we have defined the torque $\vec{\Gamma}$. Torque is an influence that causes angular acceleration, just as force is an influence that causes linear acceleration. Note that the angular momentum and torque must be calculated using the same origin.

When a planet orbits a star, it does so under the influence of gravity. Gravity exerts no torque on the planet (when the origin is chosen to be at the center of the star), hence its angular momentum is conserved. This means that both $|\vec{\ell}|$ is fixed <u>and</u> that the direction of $\vec{\ell}$ is fixed. This latter statement means that the motion of the particle is confined to a plane spanned by the $\vec{r}$ and $\vec{p}$ vectors – essentially a reduction of the problem from 3D motion to 2D motion. This allows us to use polar coordinates to describe the motion of the planet about the star. We showed that the angular momentum is $\vec{\ell} = mr^2\dot{\phi}\,\hat{z}$, where $\hat{z}$ is the direction perpendicular to the plane formed by the position and momentum vectors of the planet, with the origin in the center of the star. From this result one can show that the position vector of the planet sweeps out equal areas in equal times, $\dot{A} = \frac{1}{2}\frac{|\vec{\ell}|}{m}$, known as Kepler's second law of motion (illustrated below). The red areas show the area swept out by the position vector over equal time intervals. $A_1 = A_2$.



Kepler's Second Law:
Law of Equal Areas
(greatly exaggerated)

From http://outreach.atnf.csiro.au/education/senior/cosmicengine/renaissanceastro.html

Now consider a system of particles undergoing rotation. One can write the total angular momentum of a system of particles as $\vec{L} = \sum_{\alpha=1}^{N} \overrightarrow{r_\alpha} \times \overrightarrow{p_\alpha}$. The time rate of change of the total angular momentum vector is equal to the net external torque acting on the system: $\dot{\vec{L}} = \sum_{\alpha=1}^{N} \overrightarrow{r_\alpha} \times \overrightarrow{F_{ext}} = \vec{\Gamma}^{ext}$. This is Newton's second law of rotational motion for extended multi-particle systems. Its derivation assumes 1) all the internal forces are central in nature – they act along the line between the particles, and 2) the internal forces obey Newton's third law.

It is often convenient to write the angular momentum of a rigid body in terms of the moment of inertia as: $L_z = I_z\omega$, where the axis of rotation coincides with the z-axis and the object has angular velocity ω. You will show for homework that $I_z = \sum_{\alpha=1}^{N} m_\alpha(x_\alpha^2 + y_\alpha^2)$. Later we will attack the general problem of an arbitrary object underdoing rotation about an arbitrary axis. This will lead to introduction of the inertia tensor to relate the angular velocity vector to the angular momentum vector.

We then considered the problem (Taylor 3.35) of a solid disk rolling down an inclined plane, and solved for the acceleration of the center of mass. When considering Newton's laws applied to systems of particles, one often has to make an extended free-body diagram. In other words, instead of treating the object as a point particle with all forces applied at that point, one has to consider the extended 3D object and note the locations of the point of application of the various forces. In the rolling disk problem, the force of gravity (weight) acts at the CM (as proven above), while the normal force and static friction force are applied at the point of contact of the disk and the inclined plane. One then must choose an origin and calculate the net torque and angular moment about that origin to finally employ Newton's second law for rotational motion: $\dot{\vec{L}} = \vec{\Gamma}^{ext}$.